

Evaluating Theta Derivatives with Rational Characteristics

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Dedicated to the memory of my grandfather, Isaac Zemel

Introduction

The *Jacobi theta function* $\theta(\tau, z) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau + 2\pi i n z}$, as well as the associated *theta constants* $\theta(\tau, 0)$, are well-known for many decades to be fundamental objects in number theory. One reason is that the theta constant is a modular form of weight $\frac{1}{2}$ with respect to a congruence subgroup of low level (in fact, [SS] shows that variants of this theta function generate all the modular forms of weight $\frac{1}{2}$ with respect to every congruence subgroup), while $\theta(\tau, z)$ is the first example of a Jacobi form (see [EZ] for more on this subject). These functions also have applications in combinatorics, enumerative number theory (e.g., the theory of partitions), the theory of lattices (which yields generalizations of these theta functions), differential equations (the Jacobi theta function is a fundamental solution to the heat equation), and many other branches in mathematics.

One considers more general theta functions, involving *characteristics*. The effect of adding characteristics to the theta function boils down to translations of z and the summation index n by some real number. There are many relations between the theta functions with various characteristics (e.g., the characteristics are essentially well-defined modulo $2\mathbb{Z}$), but the substitution $z = 0$ yields many different theta constants. The theta constants with integral characteristics are very well-known modular forms of weight $\frac{1}{2}$, and the investigation of their properties goes back to the times of Jacobi and Euler. However, as integral characteristics yield only 3 functions, the number of possible constructions from them remains limited. Taking the derivative of a theta function with respect to z at $z = 0$ (these functions are called *theta derivatives*) produces a modular form of weight $\frac{3}{2}$, but with integral characteristics one obtains only one non-zero theta derivative. Moreover, the Jacobi derivative formula states that this additional function is not new, but is (up to a multiplicative constant) the product of the previous three theta constants.

It is a simple observation that by allowing the characteristics to become rational, the theta constants become modular forms (again of weight $\frac{1}{2}$) with

respect to congruence subgroups of higher level (in fact, by multiplying the argument τ by an appropriate integer and taking appropriate linear combinations one obtains all the functions considered in [SS] in this way). A similar assertion, with the weight being $\frac{3}{2}$, holds for the theta derivatives. One thus expects that a theta derivative of rational characteristics should be the product of three theta constants with rational characteristics, or a linear combinations of such products, analogously to the Jacobi derivative formula. Recently [M] developed a method to find such expressions, in which one of the multipliers has the same characteristics as the theta derivative in question and the others have integral characteristics (but with different arguments $a\tau$ for positive rational a). The idea (a small variant of which can also be used for proving the classical Jacobi derivative formula) is to evaluate the logarithmic derivative of the associated theta function at $z = 0$, when this function is expressed via the *Jacobi triple product identity* as an infinite product of simple expressions. Since the Fourier coefficients of $\theta^2(\tau, 0)$ and $\theta(\tau, 0)\theta(2\tau, 0)$ are known, using the fact that the n th coefficient is the number of presentations of n as a norm from the ring of integers in $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{-2})$ respectively (both with class number 1), it is relatively easy to recognize these series in case they arise in the evaluation of the logarithmic derivative.

Now, [M] used this method in order to evaluate 3 theta derivatives, namely those in which the upper coordinate is 1 and the lower one is $\frac{1}{2}$, $\frac{1}{4}$, or $\frac{3}{4}$. We show that using only these tools, and perhaps some simple algebraic manipulations, 19 further theta derivatives (all those in which both the characteristics are in $\frac{1}{4}\mathbb{Z}$, but at least one of them is in $\frac{1}{2}\mathbb{Z}$) can be evaluated. Moreover, there is another recognizable theta series, namely the one associated to $\mathbb{Q}(\sqrt{-3})$. Using this series we can evaluate 16 additional theta derivatives (those whose characteristics lie in $\frac{1}{3}\mathbb{Z}$). We remark that for other imaginary quadratic fields (also those with class number 1) additional technical difficulties arise. We leave this case for future investigation.

The Jacobi derivative formula also provides us with the Fourier expansion of the third power of the Dedekind eta function. [M] also uses his results (for the characteristic with $\frac{1}{2}$) for obtaining the Fourier expansion of another eta quotient of weight $\frac{3}{2}$. We indicate how some additional results in this direction as well (many of which are parallel to the main results of [LO]), though in most of the cases our method yields the Fourier expansion of a linear combination of eta quotients, rather than of a single eta quotient. Some relations between eta quotients also follow from this argument.

This paper is divided into 3 sections. Section 1 describes the general observations about theta functions that are required for applying the argument (in fact, this is done in such a way that apart from a proof for the Jacobi triple product identity, this paper is almost fully self-contained). Section 2 presents the results of [M] again, and shows how the argument generalizes for the additional stated 19 theta derivatives in which the characteristics are from $\frac{1}{4}\mathbb{Z}$. Finally, Section 3 proves the results for characteristics from $\frac{1}{3}\mathbb{Z}$.

I would like to thank H. Farkas for many interesting discussions on this

subject.

1 Theta Functions and Theta Derivatives

Given two real numbers ε and δ , the *theta functions with characteristics* $\begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}$ is a functions of two variables, τ from the *upper half-plane* \mathcal{H} (i.e., complex τ with positive imaginary part) and $z \in \mathbb{C}$. It is defined by the infinite series expansion

$$\theta\begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}(\tau, z) = \sum_{n=-\infty}^{\infty} \mathbf{e}\left[\left(n + \frac{\varepsilon}{2}\right)^2 \frac{\tau}{2} + \left(n + \frac{\varepsilon}{2}\right)\left(z + \frac{\delta}{2}\right)\right], \quad (1)$$

where $\mathbf{e}(u)$ stands, for a complex number u , for $e^{2\pi i u}$. In addition we shall be using the classical notation $q = \mathbf{e}(\tau)$ and $\zeta = \mathbf{e}(z)$ (note that some authors, including [M] on which our work is based use q for the square root of our q), while the primitive N th root of unity $\mathbf{e}(\frac{1}{N})$ will be denoted by ζ_N (no confusion with ζ should arise). The value of this function at $z = 0$ and the value of the derivative with respect to z at $z = 0$ are called the *theta constant* and *theta derivative* with characteristics $\begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}$. We are interested in explicit expressions for the derivatives in terms of the constants. We are interested in identifying theta derivatives with rational characteristics in terms of theta constants. The Fourier expansions of the results will be clear from our arguments, while each final expression will also be phrased in terms of the Dedekind eta function

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).$$

As with rational ε and δ these theta constants become, as functions of τ , modular forms of weight $\frac{1}{2}$ on some congruence subgroup of $SL_2(\mathbb{Z})$ (the same holds for η), and the derivatives have weight $\frac{3}{2}$, one naturally expects to be able to express the derivatives as linear combinations of products of three theta constants (or of eta quotients of total weight $\frac{3}{2}$).

First, the theta functions have some well-known properties, most of which follow by a simple summation index change (see, e.g., Section 1 in Chapter 2 of [FK]). Explicitly we have the pseudo-periodicity property in z with respect to the lattice determined by τ , namely

$$\theta\begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}(\tau, z + a\tau + b) = \mathbf{e}\left(\frac{b\varepsilon - a\delta}{2}\right) \zeta^{-a} q^{-a^2/2} \theta\begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}(\tau, z), \quad (2)$$

as well as the pseudo-2-periodicity in the characteristics,

$$\theta\begin{bmatrix} \varepsilon + 2a \\ \delta + 2b \end{bmatrix}(\tau, z) = \mathbf{e}\left(\frac{b\varepsilon}{2}\right) \theta\begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}(\tau, z), \quad (3)$$

both holding for integral a and b (and every $\tau \in \mathcal{H}$ and $z \in \mathbb{C}$). Allowing arbitrary real a and b , all the theta functions are related to one another via the relation

$$\theta\begin{bmatrix} \varepsilon + a \\ \delta + b \end{bmatrix}(\tau, z) = \mathbf{e}\left(\frac{a(b+\delta)}{4}\right) \zeta^{a/2} q^{a^2/8} \theta\begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}\left(\tau, z + \frac{a\tau + b}{2}\right) \quad (4)$$

(with Equation (3) following from Equations (2) and (4)), and we also have the parity relations

$$\theta\left[\begin{smallmatrix} -\varepsilon \\ -\delta \end{smallmatrix}\right](\tau, z) = \theta\left[\begin{smallmatrix} \varepsilon \\ \delta \end{smallmatrix}\right](\tau, -z), \quad \text{whence also} \quad \theta\left[\begin{smallmatrix} -\varepsilon \\ -\delta \end{smallmatrix}\right]'(\tau, z) = -\theta\left[\begin{smallmatrix} \varepsilon \\ \delta \end{smallmatrix}\right]'(\tau, -z). \quad (5)$$

Here and throughout, the prime denotes differentiation with respect to z only. Now, Equation (3) allows us to restrict attention to ε and δ in $[0, 2)$ (or equivalently in $(-1, 1]$). Integration over the boundary of a fundamental parallelogram with respect to τ shows (using equation (2)) that each theta functions with characteristics has precisely one zero (in z) in each such parallelogram. Using Equations (5) and (2) we determine this zero for the case $\varepsilon = \delta = 1$, and more generally we obtain, by Equation (4), the following lemma.

Lemma 1.1. *If ε and δ are in $(-1, 1]$ then the unique zero of $\theta\left[\begin{smallmatrix} \varepsilon \\ \delta \end{smallmatrix}\right](\tau, z)$ that is of the form $z = c\tau + d$ with c and d in $[0, 1)$ is the point $z = \frac{1-\varepsilon}{2}\tau + \frac{1-\delta}{2}$.*

Moreover, by Equation (5) we may further restrict our attention to theta functions (and constants, and derivatives) in which the characteristics $\left[\begin{smallmatrix} \varepsilon \\ \delta \end{smallmatrix}\right]$ satisfying $0 \leq \varepsilon \leq 1$ and $0 \leq \delta < 2$, where if $\varepsilon \in \{0, 1\}$ then considering only $0 \leq \delta \leq 1$ does not leave out, up to relations, any theta derivative.

The main idea of the proofs of all the following identities, following [M], is the use of the *Jacobi triple product identity*. This identity is given by the equality

$$\sum_{n=-\infty}^{\infty} x^{n^2/2} y^n = \prod_{n=1}^{\infty} (1 - x^n)(1 + x^n y)(1 + \frac{x^n}{y}),$$

holding either as formal power series, or as convergent complex expressions in case $|x| < 1$. For a simple proof of it, see [A]. Taking $x = q$ and $y = \mathbf{e}(\frac{\delta}{2})q^{\varepsilon/2}\zeta$ in the Jacobi triple product identity, we find that $\theta\left[\begin{smallmatrix} \varepsilon \\ \delta \end{smallmatrix}\right](\tau, z)$ can be expanded as

$$\mathbf{e}(\frac{\varepsilon\delta}{4})q^{\varepsilon^2/8}\zeta^{\varepsilon/2} \prod_{n=1}^{\infty} (1 - q^n)(1 + \mathbf{e}(\delta/2)q^{n-\frac{1-\varepsilon}{2}}\zeta)(1 + \mathbf{e}(-\delta/2)q^{n-\frac{1+\varepsilon}{2}}/\zeta). \quad (6)$$

The vanishing from Lemma 1.1 (in particular the vanishing of $\theta\left[\begin{smallmatrix} \varepsilon \\ \delta \end{smallmatrix}\right](\tau, 0)$) can be easily seen from Equation (6) as well, in which it occurs only at the rightmost term with $n = 1$. In fact, all the properties appearing in Equations (2), (3), (4), and (5) can be established by appropriately manipulating Equation (6).

The first relations that one can deduce from Equation (6) are expressions for the theta constants with integral characteristics in terms of the Dedekind eta function.

Proposition 1.2. *The theta constants $\theta\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right](\tau, 0)$, $\theta\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right](\tau, 0)$, and $\theta\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right](\tau, 0)$ equal $2\frac{\eta^2(2\tau)}{\eta(\tau)}$, $\frac{\eta^2(\tau/2)}{\eta(\tau)}$, and $\frac{\eta^5(\tau)}{\eta^2(2\tau)\eta^2(\tau/2)}$ respectively.*

Proof. For $\theta\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right](\tau, 0)$ the last term with $n = 1$ in Equation (6) yields the constant 2, and the rest gives the product $q^{1/8} \prod_{i=1}^{\infty} (1 - q^n)(1 + q^n)^2$. By writing $1 + q^n$ as

$\frac{1-q^{2n}}{1-q^n}$ we obtain the required expression. From $\theta\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right](\tau, 0)$ we get from Equation (6) just $\prod_{i=1}^n (1-q^n)(1-q^{n-1/2})^2$, and the fact that $\prod_{i=1}^n (1-q^{n-1/2})$ is the quotient $\prod_{i=1}^n (1-q^{n/2}) / \prod_{i=1}^n (1-q^n)$ yields the desired expression. Finally, the Equation (6) implies that $\theta\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right](\tau, 0)$ equals $\prod_{i=1}^n (1-q^n)(1+q^{n-1/2})^2$. We replace each $1+q^{n+1/2}$ by $\frac{1-q^{2n-1}}{1-q^{n-1/2}}$, and apply the same operation to $\prod_{i=1}^n (1-q^{n-1/2})$ from above as well as write $\prod_{i=1}^n (1-q^{2n-1})$ as $\prod_{i=1}^n (1-q^n) / \prod_{i=1}^n (1-q^{2n})$ (by the exact same argument), and the resulting expression follows (the external powers of q combine to the correct factors in all three cases). This proves the proposition \square

We have seen that $\theta\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right](\tau, 0)$ vanishes, and the same assertion holds for the three theta derivatives $\theta\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]'(\tau, 0)$, $\theta\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]'(\tau, 0)$, and $\theta\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]'(\tau, 0)$. The remaining integral theta derivative is given in terms of the *Jacobi derivative formula*, that reads

$$\theta\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]'(\tau, 0) = -\pi\theta\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right](\tau, 0)\theta\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right](\tau, 0)\theta\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right](\tau, 0) = -2\pi\eta^3(\tau). \quad (7)$$

By replacing the theta derivative on the left hand side of Equation (7) by the derivative of the series from Equation (1) at $z = 0$ and using the obvious symmetry between n and $-1-n$ we find that the Fourier expansion of $\eta^3(\tau)$ is $\sum_{n=1}^{\infty} (-1)^n (2n+1) q^{(2n+1)^2/8}$, or equivalently $\sum_{n=1}^{\infty} n \left(\frac{-4}{n}\right) q^{n^2/8}$ (another equivalent statement follows by taking out the external coefficient $q^{1/8}$, showing that $\prod_{n=1}^{\infty} (1-q^n)^3$ expands as $\sum_{n=1}^{\infty} (-1)^n (2n+1) q^{n(n+1)/2}$). A simple proof, which we shall indicate in the following paragraph, can be given using the Jacobi triple product identity. [M] derived similar expressions, again using the Jacobi triple product identity, for $\theta\left[\begin{smallmatrix} 1 \\ 1/2 \end{smallmatrix}\right]'(\tau, 0)$, $\theta\left[\begin{smallmatrix} 1 \\ 1/4 \end{smallmatrix}\right]'(\tau, 0)$, and $\theta\left[\begin{smallmatrix} 1 \\ 3/4 \end{smallmatrix}\right]'(\tau, 0)$ (see his Theorems 1 and 2—the later results of that reference use additional tools, that we shall not require in our investigation).

The case of the classical Jacobi derivative formula is simple: The expansion of $\theta\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right](\tau, z)$ includes the multiplier $1 - \frac{1}{\xi}$ from the rightmost term with $n = 1$ in Equation (6). Hence the derivative at $z = 0$ is just $2\pi i$ times the remaining expression evaluated at $z = 0$, which becomes just $i q^{1/8} \prod_{i=1}^n (1-q^n)^3$. This yields the rightmost expression in Equation (7), and the relation with the product of the three theta constants is immediate from Proposition 1.2. However, for other characteristics the theta constant does not vanish, and the idea of [M] is to take the *logarithmic* derivative of the expressions coming from the theta function in question via Jacobi triple product identity, and identifying it as a known weight 1 theta series (or with its expression via theta constants). Hence the classical, integral case is a bit misleading as for the argument required for all the other cases.

We therefore evaluate the logarithmic derivative of the expression from Equation (6) at $z = 0$, which yields

$$\frac{\theta\left[\begin{smallmatrix} \varepsilon \\ \delta \end{smallmatrix}\right]'(\tau, 0)}{\theta\left[\begin{smallmatrix} \varepsilon \\ \delta \end{smallmatrix}\right](\tau, 0)} = 2\pi i \left[\frac{\varepsilon}{2} + \sum_{n=1}^{\infty} \left(\frac{\mathbf{e}(\delta/2) q^{n-\frac{1-\varepsilon}{2}}}{1 + \mathbf{e}(\delta/2) q^{n-\frac{1-\varepsilon}{2}}} - \frac{\mathbf{e}(-\delta/2) q^{n-\frac{1+\varepsilon}{2}}}{1 + \mathbf{e}(-\delta/2) q^{n-\frac{1+\varepsilon}{2}}} \right) \right].$$

Recalling that $|q| < 1$ and that $0 \leq \varepsilon \leq 1$, we may expand every denominator as a geometric series, except when $\varepsilon = 1$, where the rightmost term with $n = 1$ is just a constant. By setting $\beta = -\mathbf{e}(-\delta/2)$ (which is a root of unity when $\delta \in \mathbb{Q}$) and changing the order of the terms, for later convenience, the latter expression for the theta derivative becomes

$$2\pi i \left[\frac{1+\beta}{2(1-\beta)} + \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} (\beta^l - \overline{\beta}^l) q^{ln} \right] \quad (8)$$

if $\varepsilon = 1$ (the constant is the sum of $\frac{1}{2}$ and $\frac{\beta}{1-\beta}$, and the denominator does not vanish since the case of $\delta = 1$ is already excluded), and

$$2\pi i \left[\frac{\varepsilon}{2} + \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} \left(\beta^l q^{l[n-\frac{1+\varepsilon}{2}]} - \overline{\beta}^l q^{l[n-\frac{1-\varepsilon}{2}]} \right) \right] \quad (9)$$

for $0 \leq \varepsilon < 1$. Observe that Equation (8) with $\delta = 0$ (and $\beta = -1$) also yields the vanishing of the theta derivative $\theta \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right]'(\tau, 0)$. As in Equation (9) with $\varepsilon = 0$ we get just a sum of the form $\sum_{n=1}^{\infty} \sum_{l=1}^{\infty} (\beta^l - \overline{\beta}^l) q^{l(n-1/2)}$ (up to a multiplicative constant), the vanishing of $\theta \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right]'(\tau, 0)$ and $\theta \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \right]'(\tau, 0)$ is also easily seen (since $\beta = \pm 1$ then). It remains to find out for which characteristics $\left[\begin{smallmatrix} \varepsilon \\ \delta \end{smallmatrix} \right]$ (under our restrictions) the series from Equations (8) and (9) yield a known function.

Our analysis will be based on the identification of the following weight 1 theta series:

Proposition 1.3. *The following equalities hold:*

$$\begin{aligned} (i) \quad & 1 + 4 \sum_{N=1}^{\infty} \sum_{2|d|N} \left(\frac{-1}{d} \right) q^{N/2} = \theta \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right]^2(\tau, 0). \\ (ii) \quad & 1 + 2 \sum_{N=1}^{\infty} \sum_{2|d|N} \left(\frac{-2}{d} \right) q^{N/2} = \theta \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right](\tau, 0) \theta \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right](2\tau, 0). \\ (iii) \quad & 1 + 6 \sum_{N=1}^{\infty} \sum_{d|N} \left(\frac{d}{3} \right) q^{N/2} = \theta \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right](\tau, 0) \theta \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right](3\tau, 0) + \theta \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right](\tau, 0) \theta \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right](3\tau, 0). \end{aligned}$$

The first two equalities are stated in [M], with references to [B] and [D] for proofs. The idea is simple: The imaginary quadratic fields $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{-2})$, and $\mathbb{Q}(\sqrt{-3})$ have class number 1, and unit groups of orders 4, 2, and 6 respectively. Hence the number of integral elements of a given positive norm is 4 (resp. 2, resp. 6) times the number of ideals of that given norm, and this expression is multiplicative in the norm. Determining the behavior of integral primes in the corresponding rings of integers shows that the coefficient of $q^{N/2}$ in the right hand side in the corresponding equation coincides with the number of ideals of

norm N for N a prime power, and this completes the proof. For part (iii) we also observe that the norm $a^2 - ab + b^2$ of $a + b\zeta_3$ can be written as $(a + \frac{b}{2})^2 + 3(\frac{b}{2})^2$, so that the sum with even $b = 2c$ yields the first summand, while the sum with odd $b = 2c + 1$ produces the second one.

The series from Proposition 1.3 can be recognized by the following description:

Lemma 1.4. *The following assertions hold:*

- (i) *The coefficient of $q^{N/2}$ in the series from part (i) of Proposition 1.3 is the sum over all odd positive divisors of N , in which a divisor d of N that is congruent to 1 modulo 4 contributes 4 to the coefficient while divisors in $3 + 4\mathbb{Z}$ contribute -4 . The sums $-4i \sum_{2|d|N} i^d$ and $-2i \sum_{d|N} (i^d - (-i)^d)$ also yield that value. Moreover, multiplying N by a power of 2 does not affect this coefficient.*
- (ii) *For the sum in part (ii) of Proposition 1.3, divisors of N that are congruent to 1 or 3 modulo 8 have a contribution of 2 to the coefficient of $q^{N/2}$ while those giving residues 5 or 7 modulo 8 have a contribution of -2 . The sum $-(\frac{-2}{t})\sqrt{2}i \sum_{2|d|N} (\zeta_8^{td} - \bar{\zeta}_8^{td})$, as well as $(\frac{2}{t})\sqrt{2} \sum_{2|d|N} (\zeta_8^{td} + \bar{\zeta}_8^{td})$, also give the same coefficient. These coefficients are also not changed when N is multiplied by a power of 2.*
- (iii) *Adding the coefficients of $q^{N/2}$ in the series from part (ii) of Proposition 1.3 and in half the one from (i) of that Proposition, we get a contribution of 4 from divisors in $1 + 8\mathbb{Z}$ and of -4 from those in $7 + 8\mathbb{Z}$ (the others' contributions cancel). Similarly, in the difference we get 4 from divisors with residue 3 modulo 8 and -4 from divisors having residue 5 modulo 8.*
- (iv) *In part (iii) of Proposition 1.3 the coefficient is obtained by letting divisors that are congruent to 1 modulo 3 contribute 6, while divisors in $2 + 3\mathbb{Z}$ contribute -6 . Another way to obtain this coefficient is via the sum $-2\sqrt{3}i \sum_{d|N} (\zeta_3^d - \bar{\zeta}_3^d)$. These coefficients remain invariant under multiplying N by a power of 3.*

The proof is very simple and straightforward. One just evaluates the characters and the powers of the roots of unity involved. The invariance under multiplying N by powers of the appropriate prime is also immediate from the fact that the set of divisors effectively contributing to the coefficient remains unchanged.

Recall that by decomposing the sums defining $\theta_0^{[0]}(\tau, 0)$ and $\theta_1^{[0]}(\tau, 0)$ in Equation (1) according to the parity of the summation index n one gets the simple relations

$$\theta_0^{[0]}(\tau, 0) = \theta_0^{[0]}(4\tau, 0) + \theta_1^{[1]}(4\tau, 0) = 2\theta_0^{[0]}(4\tau, 0) - \theta_1^{[0]}(\tau, 0). \quad (10)$$

This means that taking only the terms in $\theta_0^{[0]}(\tau, 0)$ having odd powers of $q^{1/2}$, or multiplying the coefficient of $q^{N/2}$ by $(-1)^N$, yields just $\theta_1^{[1]}(4\tau, 0)$ and

$\theta\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right](\tau, 0)$ respectively. We remark that Equation (10) combines with Proposition 1.2 to yield the eta identities

$$\frac{\eta^5(\tau)}{\eta^2(\frac{\tau}{2})\eta^2(2\tau)} = \frac{\eta^6(4\tau) + 2\eta^2(2\tau)\eta^4(8\tau)}{\eta^2(2\tau)\eta(4\tau)\eta^2(8\tau)} = \frac{2\eta(\tau)\eta^5(4\tau) - \eta^2(\frac{\tau}{2})\eta^2(2\tau)\eta^2(8\tau)}{\eta(\tau)\eta^2(2\tau)\eta^2(8\tau)}, \quad (11)$$

from which additional eta identities may be obtained. However, we shall not pursue eta identities further in this paper.

We now prove some useful equivalents of Equation (10) for the series from Proposition 1.3.

Corollary 1.5. *(i) Taking only the sum over odd indices N in part (i) of Proposition 1.3 yields the combination $\theta\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]^2(\tau, 0) - \theta\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]^2(2\tau, 0)$. On the other hand, by multiplying the coefficient of $q^{N/2}$ by $(-1)^N$ we obtain the expansion of $2\theta\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]^2(2\tau, 0) - \theta\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]^2(\tau, 0)$.*

(ii) Doing these operations in part (ii) Proposition 1.3 gives the combinations $\theta\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right](2\tau, 0)\theta\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right](4\tau, 0)$ and $\theta\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right](2\tau, 0)\theta\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right](\tau, 0)$ respectively.

(iii) If in the first series from part (ii), containing only odd powers of $q^{1/2}$, we multiply the coefficient of $q^{N/2}$ by $(\frac{-1}{N})$, then the resulting power series is $\theta\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right](4\tau, 0)\theta\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right](2\tau, 0)$. Doing this operation on the first series from part (i) does not affect that series.

Proof. Parts (i) and (ii) follow immediately from the invariance of the coefficients of the series in parts (i) and (ii) of Proposition 1.3 under multiplying N by a power of 2 (see parts (i) and (ii) of Lemma 1.4), using Equation (10). The second assertion from part (iii) is immediate since the power series from part (i) of Proposition 1.3 is known not to contain any powers of $q^{1/2}$ whose exponents are congruent to 3 modulo 4. For the remaining assertion we consider the residues modulo 4 of the powers appearing in the series $\theta\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right](2\tau, 0)\theta\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right](4\tau, 0)$ from part (ii) here. Since odd squares are congruent to 1 modulo 4 (and even squares are divisible by 4—this is the reason for the correctness of the second assertion of part (iii) here), when we decompose $\theta\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right](2\tau, 0)$ as in Equation (10) we see that the product $\theta\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right](8\tau, 0)\theta\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right](4\tau, 0)$ yields only powers of $q^{N/2}$ with $N \equiv 1 \pmod{4}$, while $\theta\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right](8\tau, 0)\theta\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right](4\tau, 0)$ involves only exponents in $3 + 4\mathbb{Z}$ (in fact, the same residues are obtained modulo 8 since the series from part (ii) of Proposition 1.3 does not contain any term $q^{N/2}$ in which N is congruent to 5 or 7 modulo 8). Hence inverting the sign of the coefficients that give residue 3 modulo 4 just corresponds to inverting the sign of $\theta\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right](8\tau, 0)\theta\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right](4\tau, 0)$, and another application of Equation (10) shows that this operation produces the asserted expression. This completes the proof of the corollary. \square

We shall also be needing some properties and variants of the series appearing in part (iii) of Proposition 1.3, a series that we shall denote, when convenient, also by Θ_3 .

- Corollary 1.6.** (i) *The part that is based only on odd powers of N in the series from part (iii) of Proposition 1.3 is $\Theta_3(\tau) - \Theta_3(4\tau)$. By multiplying the coefficient of $q^{N/2}$ in the series from part (iii) of Proposition 1.3 by $(-1)^N$ one obtain $2\Theta_3(4\tau) - \Theta_3(\tau)$. On the other hand, taking only powers of N that are not divisible by 3 we obtain the coefficients of $\Theta_3(\tau) - \Theta_3(3\tau)$.*
- (ii) *Restricting the sum appearing in part (iv) of Lemma 1.4, or equivalently the set of divisors under consideration, to only odd divisors of N produces the coefficients of the series $\Theta_3(\tau) + \Theta_3(2\tau)$. On the other hand, taking only those divisors $d|N$ such that $\frac{N}{d}$ is odd the series $\Theta_3(\tau) - \Theta_3(2\tau)$.*
- (iii) *Multiplying the term associated with a divisor d (again, in the sum from part (iv) of Lemma 1.4) by $(-1)^d$, we obtain the coefficient of $q^{N/2}$ in $-\Theta_3(\tau) - 2\Theta_3(2\tau)$. Doing so with $(-1)^{N/d}$, the coefficients are those of the series $+\Theta_3(\tau) - 2\Theta_3(2\tau)$. The series obtained by replacing ζ_3 in part (iv) of Lemma 1.4 by ζ_6 equals $\Theta_3(\tau) + 2\Theta_3(2\tau)$.*
- (iv) *Restricting the sum involving ζ_6 from part (iii) to odd divisors gives the same result as for ζ_3 with the same restriction. On the other hand, applying the restriction of odd $\frac{N}{d}$ produces the coefficients of the series $\Theta_3(\tau) + \Theta_3(2\tau) - 2\Theta_3(4\tau)$.*

Proof. The fact that multiplying a divisor d of N by 2 inverts the value of $\zeta_3^d - \overline{\zeta}_3^d$ (or its residue modulo 3) implies that if $2d|N$ then the contributions of d and $2d$ to the coefficient of $q^{N/2}$ in the series Θ_3 from part (iii) of Proposition 1.3 cancel. Hence if the power of 2 dividing N is even then one may consider the coefficient as arising only from the contributions arising either from odd d or from d with odd $\frac{N}{d}$, while if this power of 2 is odd then $q^{N/2}$ does not appear in $\Theta_3(\tau)$. The first assertion in part (i) is then established, and the second assertion there follows as in the proof of parts (i) and (ii) of Corollary 1.5. The third assertion of that part is immediate from the invariance of these coefficients under multiplication of N by 3 (by part (iv) of Lemma 1.4). The analysis of the powers of 2 also implies that both restrictions appearing in part (ii) do not affect the coefficient of $q^{N/2}$ if the 2-adic valuation of N is even, while for N with odd 2-adic valuation we find that restricting to odd d yields the coefficient associated with $\frac{N}{2}$, while the restriction to d with odd $\frac{N}{d}$ produces the additive inverse of the latter coefficient. This proves part (ii) as well. By observing that multiplying terms by -1 is the same as subtracting twice the corresponding series from our original Θ_3 , part (ii) implies the first two assertions of part (iii). When checking the powers of ζ_6 , we find that the term $\zeta_6^d - \overline{\zeta}_6^d$ coincides with $\zeta_3^d - \overline{\zeta}_3^d$ for odd d but with its additive inverse for even d . The second assertion in part (iv) immediately follows, and as this behavior is also described by multiplying $\zeta_3^d - \overline{\zeta}_3^d$ by $-(-1)^d$, the last assertion of part (iii) follows from the first assertion of that part. Finally, the comparison between the expressions with ζ_3 and ζ_6 combines with the proof of the second assertion of part (iii) to show that the restriction to odd $\frac{N}{d}$ yields, in the sum based on powers of ζ_6 , the

same expression as for ζ_3 if N is odd and its additive inverse when N is even. The form of the desired expression thus follows from parts (i) and (ii). This proves the corollary. \square

2 Theta Derivatives with Characteristics from $\frac{1}{4}\mathbb{Z}$

Theorem 1 of [M] identifies the theta derivative $\theta[\frac{1}{1/2}]'(\tau, 0)$ as the expression $-\pi\theta[\frac{1}{1/2}](\tau, 0)\theta[\frac{0}{0}]^2(2\tau, 0)$. Indeed, when one substitutes the values $\varepsilon = 1$ and $\delta = \frac{1}{2}$ (hence $\beta = i$) into the expression from Equation (8) and writes $N = nl$ and $d = l$, the terms with even d cancel out, the ones with odd d give $-4\pi i \cdot i^d$, and the constant coefficient $2\pi i \frac{1+i}{2(1-i)}$ equals $-\pi$. The assertion thus follows from part (i) of Lemma 1.4 (but note that the power series is with q^N rather than $q^{N/2}$). Some other theta derivatives that use that part are given in the following theorem.

Theorem 2.1. *Explicit expressions for the three theta derivatives $\theta[\frac{0}{1/2}]'(\tau, 0)$, $\theta[\frac{1/2}{1}]'(\tau, 0)$, and $\theta[\frac{1/2}{0}]'(\tau, 0)$ are given by the equalities*

$$\theta[\frac{0}{1/2}]'(\tau, 0) = \pi\theta[\frac{0}{1/2}](\tau, 0)\left(\theta[\frac{0}{0}]^2(\tau, 0) - \theta[\frac{0}{0}]^2(2\tau, 0)\right), \quad (12)$$

$$\theta[\frac{1/2}{1}]'(\tau, 0) = \frac{\pi i}{2}\theta[\frac{1/2}{1}](\tau, 0)\theta[\frac{0}{0}]^2(\frac{\tau}{2}, 0), \quad (13)$$

and

$$\theta[\frac{1/2}{0}]'(\tau, 0) = \frac{\pi i}{2}\theta[\frac{1/2}{0}](\tau, 0)\left(2\theta[\frac{0}{0}]^2(\tau, 0) - \theta[\frac{0}{0}]^2(\frac{\tau}{2}, 0)\right) \quad (14)$$

respectively. In addition, we have the equalities

$$\theta[\frac{1/2}{1/2}]'(\tau, 0) = \frac{\pi i}{2}\theta[\frac{1/2}{1/2}](\tau, 0)\left(2\theta[\frac{0}{0}]^2(2\tau, 0) - (1+i)\theta[\frac{0}{0}]^2(\tau, 0) + i\theta[\frac{0}{0}]^2(\frac{\tau}{2}, 0)\right) \quad (15)$$

and

$$\theta[\frac{1/2}{3/2}]'(\tau, 0) = \frac{\pi i}{2}\theta[\frac{1/2}{3/2}](\tau, 0)\left(2\theta[\frac{0}{0}]^2(2\tau, 0) - (1-i)\theta[\frac{0}{0}]^2(\tau, 0) - i\theta[\frac{0}{0}]^2(\frac{\tau}{2}, 0)\right). \quad (16)$$

Proof. With $\varepsilon = 0$ and $\delta = \frac{1}{2}$ (hence with $\beta = i$ again), Equation (9) takes the form $2\pi i \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} (i^l - (-i)^l) q^{l(n-1/2)}$. Again the terms with even l vanish, so that only powers $q^{N/2}$ with odd N survive. As the terms with odd l give the required coefficient of $q^{N/2}$ for such N by part (i) of Lemma 1.4, the Equation (12) follows from part (i) of Corollary 1.5. For the Equation (13) we set $\varepsilon = \frac{1}{2}$ and $\delta = 1$ (so that $\beta = 1$) in Equation (9) and get a constant coefficient of $\frac{\pi i}{2}$, terms of the form $q^{l(4n-3)/4}$ with the coefficient $2\pi i$, and terms $q^{l(4n-1)/4}$ multiplied by $-2\pi i$. Writing the exponent of q as $\frac{N}{4}$ and taking the divisor d

to be $4n - 1$ or $4n - 3$ (positive, with residues 3 and 1 modulo 4 respectively), we obtain again a description as in part (i) of Lemma 1.4, proving the desired result. Now replace δ from 1 to 0 hence β from 1 to -1 in the latter argument, so that each term involving an index l is multiplied by $(-1)^l$. Since N is l times an odd number, this sign is the same as $(-1)^N$, and Equation (14) thus follows from part (i) of Corollary 1.5.

Turning to the case where $\varepsilon = \frac{1}{2}$ and δ is either $\frac{1}{2}$ or $\frac{3}{2}$, a parameter that we shall write as $\frac{2\mp 1}{2}$, we find that $\beta = \pm i$. An argument similar to the previous paragraph yields the same constant coefficient, while now $q^{l(4n-3)/4}$ comes multiplied by $2\pi i(\pm i)^l$ and $q^{l(4n-1)/4}$ has a multiplier of $-2\pi i(\mp i)^l$. The terms with even l (or equivalently even N) can be analyzed just like the case of characteristics $\begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$ (but with the variable multiplied by 2), thus yielding the common terms in Equations (15) and (16). For odd N we can take the sign out of the powers, and obtain a coefficient of $\pm 2\pi i \cdot i^l$ in front of both terms $q^{l(4n-3)/4}$ and $q^{l(4n-1)/4}$. Writing N for the exponent and $d = l$, we obtain from part (i) of Lemma 1.4 that this is $\mp \frac{\pi}{2}$ times the part of the series from part (i) of Proposition 1.3 involving only odd powers of $q^{1/4}$, which yields the remaining terms by part (i) of Corollary 1.5. This completes the proof of the theorem. \square

We recall that Equation (8) of [M] applies Theorem 1 of that reference for obtaining the Fourier expansion of the eta quotient $\frac{\eta^9(2\tau)}{\eta^3(\tau)\eta^3(4\tau)}$. The idea is that arguments similar to the proof of Proposition 1.2 (i.e., using the Jacobi triple product identity) and the fact that the product of $1 - iq^n$ and $1 + iq^n$ is $1 + q^{2n}$ show that $\theta[\frac{1}{1/2}](\tau, 0)$ is the theta quotient $\sqrt{2} \frac{\eta(\tau)\eta(4\tau)}{\eta(2\tau)}$ (the leading coefficient is the product of ζ_8 and $1 - i$). Combining this with Proposition 1.2 itself and the derivative of Equation (1) (together with some summation index change) yields the desired equality of the eta quotient (without the leading coefficient) to the Fourier series $\sum_{n=0}^{\infty} n(\frac{-2}{n})q^{n^2/8}$. However, we can do the same analysis also using Equations (12), (13), and (14).

Corollary 2.2. *The following combinations of eta quotients can be expanded as explicit Fourier series:*

$$\frac{\eta^{14}(2\tau)\eta^4(\frac{\tau}{2}) - \eta^{14}(\tau)\eta^4(4\tau)}{\eta^4(\frac{\tau}{2})\eta^4(\tau)\eta^2(2\tau)\eta^5(4\tau)} = 4 \sum_{n=0}^{\infty} n(\frac{-4}{n})q^{n^2/2},$$

$$\frac{\eta^9(\frac{\tau}{2})}{\eta^3(\frac{\tau}{4})\eta^3(\tau)} = \sum_{n=0}^{\infty} n(\frac{-2}{n})q^{n^2/32},$$

and

$$\frac{2\eta^{14}(\tau)\eta^4(\frac{\tau}{4}) - \eta^{14}(\frac{\tau}{2})\eta^4(2\tau)}{\eta^5(\frac{\tau}{4})\eta^2(\frac{\tau}{2})\eta^4(\tau)\eta^4(2\tau)} = \sum_{n=0}^{\infty} n(\frac{-4}{n})q^{n^2/32}.$$

Proof. Analyzing the triple product expansions of $\theta[\frac{0}{1/2}](\tau, 0)$, $\theta[\frac{1/2}{1}](\tau, 0)$, and $\theta[\frac{1/2}{0}](\tau, 0)$ from Equation (6) by means similar to the ones from the

proof of Proposition 1.2 and Equation (8) of [M] yield the eta quotients $\frac{\eta^2(2\tau)}{\eta(4\tau)}$, $\zeta_8 \frac{\eta(\tau)\eta(\tau/4)}{\eta(\tau/2)}$, and $\frac{\eta^2(\tau/2)}{\eta(\tau/4)}$ respectively. The multipliers of these theta constants appearing in Equations (12), (13), and (14) are evaluated using Proposition 1.2, while the Fourier series of the left hand sides of these equations can be manipulated as in the proof of Equation (8) of [M]. This proves the corollary. \square

Remark 2.3. Proposition 1.2 and the proof of Corollary 2.2 show that the theta constants $\theta\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right](\tau, 0)$ and $\theta\left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right](\tau, 0)$ coincide with $\theta\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right](4\tau, 0)$ and $\theta\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right](\frac{\tau}{4}, 0)$ respectively. In addition, theta constant $\theta\left[\begin{smallmatrix} 1 \\ 1/2 \end{smallmatrix}\right](\tau, 0)$ from Theorem 1 of [M] coincides with the theta constant $\frac{\bar{\zeta}_8}{\sqrt{2}}\theta\left[\begin{smallmatrix} 1/2 \\ 1 \end{smallmatrix}\right](4\tau, 0)$ from the proof of that Corollary. Unfortunately, the theta constants appearing in Equations (15) and (16) (as well as in all the formulae from Theorems 2.4 and 2.5) cannot be expressed as eta quotients via such an argument. We also remark that the theta constants and theta derivatives that can be expressed as single eta quotients (not linear combinations) are closely related to some of the expansions appearing in Theorems 1.1 and 1.2 in [LO]. We shall encounter another few such equalities below (with additional relations to [LO]), though involving only non-integral characteristics. Moreover, the second equality in Corollary 2.2 is just Equation (8) of [M] (after a change of variable). On the other hand, the other two equations in that Corollary can be compared with the expansions of $4\eta^3(4\tau)$ and $\eta^3(\frac{\tau}{4})$ arising from Equation (7) yielding another two eta identities, which may or may not follow from the eta identities in Equation (11) (this point requires further investigation).

Theorem 2 of [M] concerns the theta derivative $\theta\left[\begin{smallmatrix} 1 \\ (2\pm 1)/4 \end{smallmatrix}\right]'(\tau, 0)$ (namely $\theta\left[\begin{smallmatrix} 1 \\ 1/4 \end{smallmatrix}\right]'(\tau, 0)$ and $\theta\left[\begin{smallmatrix} 1 \\ 3/4 \end{smallmatrix}\right]'(\tau, 0)$), asserting that it can be expressed as

$$-\pi\theta\left[\begin{smallmatrix} 1 \\ (2\pm 1)/4 \end{smallmatrix}\right](\tau, 0)\theta\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right](4\tau, 0)\left(\sqrt{2}\theta\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right](2\tau, 0) \pm \theta\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right](4\tau, 0)\right).$$

The proof goes along the same lines (just expressed a bit differently in [M]): β is $\zeta_8^{2\pm 1}$, so that Equation (8) yields a constant coefficient of $-\pi(\sqrt{2} \pm 1)$ (this is the value of $\pi i \frac{1+\zeta_8^t}{1-\zeta_8^t}$ for t being 3 resp. 1, since $(1-\zeta_8)(1-\zeta_8^3) = -\sqrt{2}i$), while q^N with positive N comes with the coefficient $2\pi i \sum_{l|N} (\zeta_8^{(2\pm 1)l} - \bar{\zeta}_8^{(2\pm 1)l})$. The terms in which l is odd combine, by part (ii) of Lemma 1.4, to the first term preceding the \pm (since both $t = 1$ and $t = 3$ satisfy $(\frac{-2}{t}) = +1$), while part (i) of that Lemma implies the those with even l produce the other term (as in the proof of Theorem 2.1). However, several other theta derivatives can be determined in this manner.

Theorem 2.4. *For the theta derivatives $\theta\left[\begin{smallmatrix} 0 \\ 1/4 \end{smallmatrix}\right]'(\tau, 0)$ and $\theta\left[\begin{smallmatrix} 0 \\ 3/4 \end{smallmatrix}\right]'(\tau, 0)$ we have the expansions*

$$-\pi\theta\left[\begin{smallmatrix} 0 \\ 1/4 \end{smallmatrix}\right](\tau, 0)\left(\sqrt{2}\theta(2\tau, 0)\tilde{\theta}(4\tau, 0) - \theta^2(2\tau, 0) + \theta^2(4\tau, 0)\right) \quad (17)$$

and

$$-\pi\theta\left[\frac{0}{3/4}\right](\tau, 0)\left(\sqrt{2}\theta(2\tau, 0)\tilde{\theta}(4\tau, 0) + \theta^2(2\tau, 0) - \theta^2(4\tau, 0)\right) \quad (18)$$

respectively, where θ stands for $\theta\left[\frac{0}{0}\right]$ and $\tilde{\theta}$ denotes $\theta\left[\frac{1}{0}\right]$. Denoting, in addition, $\theta\left[\frac{0}{1}\right]$ by $\hat{\theta}$, we find that the respective expressions of the theta derivatives $\theta\left[\frac{1/2}{1/4}\right]'(\tau, 0)$, $\theta\left[\frac{1/2}{3/4}\right]'(\tau, 0)$, $\theta\left[\frac{1/2}{5/4}\right]'(\tau, 0)$, and $\theta\left[\frac{1/2}{7/4}\right]'(\tau, 0)$ are given by the formulae

$$\frac{\pi i}{2}\theta\left[\frac{1/2}{1/4}\right](\tau, 0)\left[2\theta^2(4\tau, 0) - (1-i)\theta^2(2\tau, 0) - i\theta^2(\tau, 0) - \sqrt{2}\tilde{\theta}(2\tau, 0)\left(\hat{\theta}(\tau, 0) - i\theta(\tau, 0)\right)\right], \quad (19)$$

$$\frac{\pi i}{2}\theta\left[\frac{1/2}{3/4}\right](\tau, 0)\left[2\theta^2(4\tau, 0) - (1+i)\theta^2(2\tau, 0) + i\theta^2(\tau, 0) + \sqrt{2}\tilde{\theta}(2\tau, 0)\left(\hat{\theta}(\tau, 0) + i\theta(\tau, 0)\right)\right], \quad (20)$$

$$\frac{\pi i}{2}\theta\left[\frac{1/2}{5/4}\right](\tau, 0)\left[2\theta^2(4\tau, 0) - (1-i)\theta^2(2\tau, 0) - i\theta^2(\tau, 0) + \sqrt{2}\tilde{\theta}(2\tau, 0)\left(\hat{\theta}(\tau, 0) - i\theta(\tau, 0)\right)\right], \quad (21)$$

and

$$\frac{\pi i}{2}\theta\left[\frac{1/2}{7/4}\right](\tau, 0)\left[2\theta^2(4\tau, 0) - (1+i)\theta^2(2\tau, 0) + i\theta^2(\tau, 0) - \sqrt{2}\tilde{\theta}(2\tau, 0)\left(\hat{\theta}(\tau, 0) + i\theta(\tau, 0)\right)\right]. \quad (22)$$

Proof. We write the first two characteristics as $\left[\frac{0}{(2\pm 1)/4}\right]$, and substitute $\varepsilon = 0$ and $\delta = \frac{2\pm 1}{4}$ (whence $\beta = \zeta_8^{2\pm 1}$ as in Theorem 2 of [M]) into Equation (9). This yields no constant term, and a coefficient of $2\pi i \sum_{d|N_2} (\zeta_8^{(2\pm 1)2^a d} - \bar{\zeta}_8^{(2\pm 1)2^a d})$ in front of $q^{N/2}$ where N is a positive integer decomposing as $2^a N_2$ for some $a \in \mathbb{N}$ (which equals the 2-adic valuation of N) and some odd number N_2 . Indeed, our N is $l(2n-1)$, so that $\frac{N}{l}$ must be odd and l is of the form $2^a d$ for some $d|N_2$. The same argument from the proof of Theorem 2 of [M] given above shows that for odd N this is just $-\sqrt{2}\pi$ times the coefficient appearing in part (ii) of Lemma 1.4 (regardless of the sign \pm), but since we have these coefficients only for odd N , part (ii) of Corollary 1.5 implies that this part of the expansion equals the first term in Equations (17) and (18). The coefficient in front of $q^{N/2}$ vanishes if $4|N$, while if $N \equiv 2 \pmod{4}$ we get from part (i) of Lemma 1.4 that this is $\mp \pi$ times the coefficient of $q^{N/2}$ in the series from part (i) of Proposition 1.3. But since only odd powers $\frac{N}{2}$ have to be taken, we apply part (i) of Corollary 1.5 again, which yields the last two terms in Equations (17) and (18).

We now turn to the remaining 4 characteristics, which we write as $\left[\frac{1/2}{(12-t)/4}\right]$ for t being 5, and 7, 9, or 11. Then β is just ζ_8^t , and the substitution of $\varepsilon = \frac{1}{2}$ and this value of β in the expression appearing in Equation (9) yields the series with the constant term $\frac{\pi i}{2}$, and with terms of the forms $2\pi i \zeta_8^{tl} q^{l(4n-3)/4}$ and $-2\pi i \bar{\zeta}_8^{tl} q^{l(4n-1)/4}$. Once again we look for the coefficient of $q^{N/4}$, and the parity of N coincides with that of l . We first consider the terms associated with even indices l (hence also N), where by writing $l = 2m$ and $\zeta_8^{2t} = \pm i$ one easily sees that the proof of Equations (15) and (16) in Theorem 2.1 transforms this

part to the first three terms in Equations (19), (20), (21), and (22). Indeed, the description of the coefficients is the same (but with powers of $q^{1/2}$), and one just notes that here the characteristics $[\frac{1/2}{3/4}]$ and $[\frac{1/2}{7/4}]$ (where t is 9 or 5) yield the $+$ sign, while with $[\frac{1/2}{1/4}]$ and $[\frac{1/2}{5/4}]$ (i.e., if t is 11 or 7) the sign is $-$. For odd N we write the coefficient $2\pi i \zeta_8^{tl}$ arising from a divisor l of N with $\frac{N}{l} \equiv 1 \pmod{4}$ as the sum of $\pi i(\zeta_8^{tl} - \bar{\zeta}_8^{tl})$ and $\pi i(\zeta_8^{tl} + \bar{\zeta}_8^{tl})$, while for divisors with $\frac{N}{l} \equiv 1 \pmod{4}$ the coefficient $-2\pi i \bar{\zeta}_8^{tl}$ equals the difference between $\pi i(\zeta_8^{tl} - \bar{\zeta}_8^{tl})$ and $\pi i(\zeta_8^{tl} + \bar{\zeta}_8^{tl})$. Taking the first terms from both types of divisors yields, for each value of t , the term containing i in the rightmost brackets in all the four Equations (19), (20), (21), and (22), as follows from part (ii) of Lemma 1.4 and part (ii) of Corollary 1.5 (together with the appropriate sign coming from the value of t). On the other hand, the multiplier $(\frac{-1}{N/l})$ differentiating between the two types of divisors is the product of $(\frac{-1}{N})$ and $(\frac{-1}{l})$, and by combining part (ii) of Lemma 1.4 again with part (iii) of Corollary 1.5 we get the remaining terms in the four equations in question. This completes the proof of the theorem. \square

Some theta derivatives in which ε is $\frac{1}{4}$ or $\frac{3}{4}$ can also be evaluated using this method:

Theorem 2.5. *With θ denoting $\theta[\frac{0}{0}]$ as in Theorem 2.4, the equalities*

$$\theta[\frac{1/4}{1}]'(\tau, 0) = \frac{\pi i}{4} \theta[\frac{1/4}{1}](\tau, 0) \theta(\frac{\tau}{4}, 0) \left(2\theta(\frac{\tau}{2}, 0) - \theta(\frac{\tau}{4}, 0) \right) \quad (23)$$

and

$$\theta[\frac{3/4}{1}]'(\tau, 0) = \frac{\pi i}{4} \theta[\frac{3/4}{1}](\tau, 0) \theta(\frac{\tau}{4}, 0) \left(2\theta(\frac{\tau}{2}, 0) + \theta(\frac{\tau}{4}, 0) \right) \quad (24)$$

hold. Moreover, using the notation $\hat{\theta}$ for $\theta[\frac{0}{1}]$ again, we have

$$\theta[\frac{1/4}{0}]'(\tau, 0) = \frac{\pi i}{4} \theta[\frac{1/4}{0}](\tau, 0) \left(2\theta(\frac{\tau}{2}, 0) \hat{\theta}(\frac{\tau}{4}, 0) - 2\theta^2(\frac{\tau}{2}, 0) + \theta^2(\frac{\tau}{4}, 0) \right) \quad (25)$$

and

$$\theta[\frac{3/4}{0}]'(\tau, 0) = \frac{\pi i}{4} \theta[\frac{3/4}{0}](\tau, 0) \left(2\theta(\frac{\tau}{2}, 0) \hat{\theta}(\frac{\tau}{4}, 0) + 2\theta^2(\frac{\tau}{2}, 0) - \theta^2(\frac{\tau}{4}, 0) \right). \quad (26)$$

The additional four theta derivatives $\theta[\frac{1/4}{1/2}]'(\tau, 0)$, $\theta[\frac{1/4}{3/2}]'(\tau, 0)$, $\theta[\frac{3/4}{1/2}]'(\tau, 0)$, and $\theta[\frac{3/4}{3/2}]'(\tau, 0)$ are expressed, respectively, by the formulae

$$\frac{\pi i}{4} \theta[\frac{1/4}{1/2}](\tau, 0) \left[2\hat{\theta}(\frac{\tau}{2}, 0) \left(\theta(\tau, 0) - i\bar{\theta}(\frac{\tau}{4}, 0) \right) - 2\theta^2(\tau, 0) + (1-i)\theta^2(\frac{\tau}{2}, 0) + i\theta^2(\frac{\tau}{4}, 0) \right], \quad (27)$$

$$\frac{\pi i}{4} \theta[\frac{1/4}{3/2}](\tau, 0) \left[2\hat{\theta}(\frac{\tau}{2}, 0) \left(\theta(\tau, 0) + i\bar{\theta}(\frac{\tau}{4}, 0) \right) - 2\theta^2(\tau, 0) + (1+i)\theta^2(\frac{\tau}{2}, 0) - i\theta^2(\frac{\tau}{4}, 0) \right], \quad (28)$$

$$\frac{\pi i}{4}\theta\left[\frac{3/4}{1/2}\right](\tau, 0)\left[2\hat{\theta}\left(\frac{\tau}{2}, 0\right)\left(\theta(\tau, 0)+i\tilde{\theta}\left(\frac{\tau}{4}, 0\right)\right)+2\theta^2(\tau, 0)-(1+i)\theta^2\left(\frac{\tau}{2}, 0\right)+i\theta^2\left(\frac{\tau}{4}, 0\right)\right], \quad (29)$$

and

$$\frac{\pi i}{4}\theta\left[\frac{3/4}{3/2}\right](\tau, 0)\left[2\hat{\theta}\left(\frac{\tau}{2}, 0\right)\left(\theta(\tau, 0)-i\tilde{\theta}\left(\frac{\tau}{4}, 0\right)\right)+2\theta^2(\tau, 0)-(1-i)\theta^2\left(\frac{\tau}{2}, 0\right)-i\theta^2\left(\frac{\tau}{4}, 0\right)\right]. \quad (30)$$

Proof. With $\varepsilon = \frac{1}{4}$ (resp. $\varepsilon = \frac{3}{4}$) and $\delta = 1$ (i.e., $\beta = 1$) Equation (9) contains the constant coefficient $\frac{\pi i}{4}$ (resp. $\frac{3\pi i}{4}$), together with the terms $2\pi i q^{l(8n-5)/8}$ (resp. $2\pi i q^{l(8n-7)/8}$) and $-2\pi i q^{l(8n-3)/8}$ (resp. $-2\pi i q^{l(8n-1)/8}$) with positive natural n and l . After using the usual notation of the q -power as $q^{N/8}$, part (iii) of Lemma 1.4 proves the Equations (23) and (24). Replacing the value of δ to 0 (hence of β by -1) results in multiplying any term with index l by $(-1)^l$, which is the same as $(-1)^N$ since $\frac{N}{7}$ is odd. Combining what we just evaluated with parts (i) and (ii) of Corollary 1.5 yields the expressions for the theta derivatives considered in Equations (25) and (26).

We now turn to the remaining four theta derivatives, where δ is $\frac{1}{2}$ or $\frac{3}{2}$, hence $\beta = \pm i$. Hence in the series from Equation (9), any q -power involving l is now multiplied by a sign times $2\pi i(\pm i)^l$. As in the proof of the Equations (19), (20), (21), and (22) in Theorem 2.4, we separate this series according to the parity of l (or equivalently N), and observe that the part with even l yield expressions that we have just evaluated in Equations (25) and (26) (but in which we now take powers of $q^{1/4}$ instead of $q^{1/8}$). This yields the terms not involving i inside the parentheses in each of the Equations (27), (28), (29), and (30). For the terms with odd l we note that the coefficient $-(\mp i)^l$ is the same as $(\pm i)^l$, so that for odd N we have to take the sum over all the relevant divisors l of N (i.e., in which $\frac{N}{7}$ congruent to 3 or to 5 modulo 8 if $\varepsilon = \frac{1}{4}$, and for which $\frac{N}{7}$ has residue 1 or 7 modulo 8 if $\varepsilon = \frac{3}{4}$) of $2\pi i(\pm i)^l$. But this is easily seen to be $\pm \frac{\pi}{2}\left(\frac{-1}{N}\right)$ times the difference appearing in part (iii) of Lemma 1.4 if $\varepsilon = \frac{1}{4}$ and $\mp \frac{\pi}{2}\left(\frac{-1}{N}\right)$ times the sum from that part in case $\varepsilon = \frac{3}{4}$. Recalling the external coefficient and that we have here only odd N and the sign $\left(\frac{-1}{N}\right)$, the remaining terms in Equations (27), (28), (29), and (30) are obtained via part (iii) of Corollary 1.5 (with the variable τ divided by 4 since we work with powers of $q^{1/8}$ here). This proves the theorem. \square

3 Theta Derivatives with Characteristics from $\frac{1}{3}\mathbb{Z}$

[M] also quotes an identity from [F] (numbered as Equation (2) in [M]) involving the theta derivative with characteristics $\left[\frac{1}{1/3}\right]$. Using the properties and characterizations of the series from part (iii) of Proposition 1.3 we can prove the following relations.

Theorem 3.1. *With integral ε and $\delta = \frac{1}{3}$ we have*

$$\theta\left[\begin{smallmatrix} 1 \\ 1/3 \end{smallmatrix}\right]'(\tau, 0) = -\frac{\pi}{\sqrt{3}}\theta\left[\begin{smallmatrix} 1 \\ 1/3 \end{smallmatrix}\right](\tau, 0)\Theta_3(2\tau) \quad (31)$$

and

$$\theta\left[\begin{smallmatrix} 0 \\ 1/3 \end{smallmatrix}\right]'(\tau, 0) = -\frac{\pi}{\sqrt{3}}\theta\left[\begin{smallmatrix} 0 \\ 1/3 \end{smallmatrix}\right](\tau, 0)\left(\Theta_3(\tau) - \Theta_3(2\tau)\right), \quad (32)$$

while if δ is integral and $\varepsilon = \frac{1}{3}$ the theta derivatives are

$$\theta\left[\begin{smallmatrix} 1/3 \\ 1 \end{smallmatrix}\right]'(\tau, 0) = \frac{\pi i}{3}\theta\left[\begin{smallmatrix} 1/3 \\ 1 \end{smallmatrix}\right](\tau, 0)\Theta_3\left(\frac{2\tau}{3}\right) \quad (33)$$

and

$$\theta\left[\begin{smallmatrix} 1/3 \\ 0 \end{smallmatrix}\right]'(\tau, 0) = \frac{\pi i}{3}\theta\left[\begin{smallmatrix} 1/3 \\ 0 \end{smallmatrix}\right](\tau, 0)\left(2\Theta_3\left(\frac{4\tau}{3}\right) - \Theta_3\left(\frac{2\tau}{3}\right)\right). \quad (34)$$

Going over to characteristics in which $\delta = \frac{2}{3}$, we find that

$$\theta\left[\begin{smallmatrix} 1 \\ 2/3 \end{smallmatrix}\right]'(\tau, 0) = -\frac{\pi}{\sqrt{3}}\theta\left[\begin{smallmatrix} 1 \\ 2/3 \end{smallmatrix}\right](\tau, 0)\left(2\Theta_3(4\tau) + \Theta_3(2\tau)\right) \quad (35)$$

and

$$\theta\left[\begin{smallmatrix} 0 \\ 2/3 \end{smallmatrix}\right]'(\tau, 0) = -\frac{\pi}{\sqrt{3}}\theta\left[\begin{smallmatrix} 0 \\ 2/3 \end{smallmatrix}\right](\tau, 0)\left(\Theta_3(\tau) + \Theta_3(2\tau) - 2\Theta_3(4\tau)\right). \quad (36)$$

On the other hand, when $\varepsilon = \frac{2}{3}$ we obtain

$$\theta\left[\begin{smallmatrix} 2/3 \\ 1 \end{smallmatrix}\right]'(\tau, 0) = \frac{\pi i}{3}\theta\left[\begin{smallmatrix} 2/3 \\ 1 \end{smallmatrix}\right](\tau, 0)\left(\Theta_3\left(\frac{\tau}{3}\right) + \Theta_3\left(\frac{2\tau}{3}\right)\right) \quad (37)$$

and

$$\theta\left[\begin{smallmatrix} 2/3 \\ 0 \end{smallmatrix}\right]'(\tau, 0) = \frac{\pi i}{3}\theta\left[\begin{smallmatrix} 2/3 \\ 0 \end{smallmatrix}\right](\tau, 0)\left(2\Theta_3\left(\frac{4\tau}{3}\right) - \Theta_3\left(\frac{\tau}{3}\right) + \Theta_3\left(\frac{2\tau}{3}\right)\right). \quad (38)$$

Proof. Setting $\delta = \frac{1}{3}$ (hence $\beta = \zeta_3$) in Equation (8) yields the series with constant $\pi i \frac{1+\zeta_3}{1-\zeta_3} = -\frac{\pi}{\sqrt{3}}$ (recall that $1 + \zeta_3 = \zeta_6$ and $\Im \bar{\zeta}_6 = -\frac{\sqrt{3}}{2}$) and in which the coefficient in front of q^N is $2\pi i \sum_{l|N} (\zeta_3^l - \bar{\zeta}_3^l)$. Equation (31) thus follows from part (iv) of Lemma 1.4. Replacing the value of ε to be 0 and using Equation (9), we obtain that $q^{N/2}$ now comes with the coefficient arising as $2\pi i$ times the sum of $\zeta_3^l - \bar{\zeta}_3^l$ over divisors l of N for which $\frac{N}{l}$ is odd. Part (ii) of Corollary 1.6 then establishes Equation (32). Next, set $\varepsilon = \frac{1}{3}$ and $\delta = 1$ (i.e., $\beta = 1$) in Equation (9), we get the constant $\frac{\pi i}{3}$ together with the terms $2\pi i q^{l(3n-2)/3}$ and $-2\pi i q^{l(3n-1)/3}$ for natural n and l . Part (iv) of Lemma 1.4 then implies Equation (33). Taking the value of δ to be 0 (so that $\beta = -1$), the expressions just mentioned above are multiplied by $(-1)^l$, which is $(-1)^{N/d}$ if d is the divisor $3n-2$ or $3n-1$ considered in the sum from part (iv) of Lemma 1.4. Equation (34) thus follows from part (iii) of Corollary 1.6.

We now substitute $\delta = \frac{2}{3}$ (which implies $\beta = \zeta_6$) in Equation (8). The constant $\pi i \frac{1+\zeta_6}{1-\zeta_6}$ becomes $-\sqrt{3}\pi$ (as $1-\zeta_6 = \bar{\zeta}_6$ and $\zeta_3+\zeta_6 = i\sqrt{3}$), and q^N comes multiplied by $2\pi i \sum_{l|N} (\zeta_6^l - \bar{\zeta}_6^l)$. Part (iii) of Corollary 1.6 implies Equation (35)

(the constant term is also correct since the constant term of $2\Theta_3(4\tau) + \Theta_3(2\tau)$ is 3 times the one of Θ_3 itself). Leaving the value of δ untouched, but working with $\varepsilon = 0$ in Equation (9), we find that the coefficient of $q^{N/2}$ is the sum involving ζ_6 , but restricted to divisors d with $\frac{N}{d}$ odd. Here we apply part (iv) of Corollary 1.6 to obtain Equation (36). Turning to the case where $\varepsilon = \frac{2}{3}$ and $\delta = \beta = 1$, the series from Equation (9) consists of the constant term $\frac{2\pi i}{3}$ as well as the terms $2\pi i q^{l(6n-5)/6}$ and $-2\pi i q^{l(6n-1)/6}$. As this is the usual sum from part (iv) of Lemma 1.4 (with the power of q divided by 3) but with the divisors being restricted to be odd, part (ii) of Corollary 1.6 proves Equation (37). Finally, with $\delta = 0$ and $\beta = -1$ (and the same value of ε) we get the same expressions but multiplied by $(-1)^l$. As the divisors $6n - 5$ or $6n - 1$ of N are odd, this sign is the same as $(-1)^N$, so that Equation (38) follows from part (i) of Corollary 1.6 and what we have just proved. This completes the proof of the theorem. \square

Remark 3.2. The theta constants appearing in all the equations in Theorem 3.1 can be expressed as eta quotients, using again the arguments from Proposition 1.2 and Corollary 2.2, together with equalities like $(1 \pm \zeta_3 q^n)(1 \pm \bar{\zeta}_3 q^n) = \frac{1 \pm q^{3n}}{1 \pm q^n}$. The resulting expressions for the theta constants from Equations (31), (32), (33), (34), (35), (36), (37), and (38) are $\sqrt{3}\eta(3\tau)$, $\frac{\eta^2(\tau)\eta(3\tau/2)}{\eta(\tau/2)\eta(3\tau)}$, $\zeta_{12}\eta(\frac{\tau}{3})$, $\frac{\eta^2(\tau)\eta(2\tau/3)}{\eta(2\tau)\eta(\tau/3)}$, $\frac{\eta^2(\tau)\eta(6\tau)}{\eta(2\tau)\eta(3\tau)}$, $\frac{\eta^2(3\tau)\eta(2\tau)\eta(\tau/2)}{\eta(\tau)\eta(3\tau/2)\eta(6\tau)}$, $\zeta_6 \frac{\eta^2(\tau)\eta(\tau/6)}{\eta(\tau/2)\eta(\tau/3)}$, and $\frac{\eta^2(\tau/3)\eta(2\tau)\eta(\tau/2)}{\eta(\tau)\eta(2\tau/3)\eta(\tau/6)}$ respectively. In particular one finds the equality of $\theta[\frac{1}{1/3}](\tau, 0)$ and $\bar{\zeta}_{12}\sqrt{3}\theta[\frac{1}{1}](9\tau, 0)$, already appearing implicitly in [F] and in Equation (2) of [M]. Moreover, the Fourier expansions of both the associated theta constants and the corresponding theta derivatives (both from Equation (1)) reduce, in many of these cases, to simple expressions. This relates to some of the equalities appearing in Theorem 1.1 of [LO] as well. However, as the expression $\frac{\eta^5(\tau)\eta^5(3\tau)}{\eta^2(\tau/2)\eta^2(2\tau)\eta^2(3\tau/2)\eta^2(6\tau)} + 4\frac{\eta^2(2\tau)\eta^2(6\tau)}{\eta(\tau)\eta(3\tau)}$ for $\Theta_3(\tau)$ is not a single eta quotient, we do not obtain a simple expression in terms of eta quotients for any of the theta derivatives from Theorem 3.1. For the theta functions considered in Theorem 3.3 below, also the theta constants themselves do not produce simple eta quotients by this argument.

The information that we have about Θ_3 suffices for obtaining formulae for yet another 8 theta derivatives.

Theorem 3.3. *We have the equalities*

$$\theta[\frac{1}{1/3}]'(\tau, 0) = -\frac{\pi i}{3}\theta[\frac{1}{1/3}](\tau, 0)\left(\bar{\zeta}_3\Theta_3(\frac{2\tau}{3}) + i\zeta_3\sqrt{3}\Theta_3(2\tau)\right) \quad (39)$$

and

$$\theta[\frac{1}{5/3}]'(\tau, 0) = -\frac{\pi i}{3}\theta[\frac{1}{5/3}](\tau, 0)\left(\zeta_3\Theta_3(\frac{2\tau}{3}) - i\bar{\zeta}_3\sqrt{3}\Theta_3(2\tau)\right), \quad (40)$$

while $\theta[\frac{1}{2/3}]'(\tau, 0)$ and $\theta[\frac{1}{4/3}]'(\tau, 0)$ are

$$-\frac{\pi i}{3}\theta[\frac{1}{2/3}](\tau, 0)\left(2\bar{\zeta}_3\Theta_3(\frac{4\tau}{3}) - \zeta_3\Theta_3(\frac{2\tau}{3}) + 2i\zeta_3\sqrt{3}\Theta_3(4\tau) + i\bar{\zeta}_3\sqrt{3}\Theta_3(2\tau)\right) \quad (41)$$

and

$$-\frac{\pi i}{3}\theta\left[\frac{1}{3}\right](\tau, 0)\left(2\zeta_3\Theta_3\left(\frac{4\tau}{3}\right)-\bar{\zeta}_3\Theta_3\left(\frac{2\tau}{3}\right)-2i\bar{\zeta}_3\sqrt{3}\Theta_3(4\tau)-i\zeta_3\sqrt{3}\Theta_3(2\tau)\right) \quad (42)$$

respectively. On the other hand, the respective expressions for $\theta\left[\frac{2}{3}\right]'(\tau, 0)$ and $\theta\left[\frac{2}{5/3}\right]'(\tau, 0)$ are

$$-\frac{\pi i}{3}\theta\left[\frac{2}{3}\right](\tau, 0)\left(\bar{\zeta}_3\Theta_3\left(\frac{\tau}{3}\right)+\zeta_3\Theta_3\left(\frac{2\tau}{3}\right)+i\zeta_3\sqrt{3}\Theta_3(\tau)-i\bar{\zeta}_3\sqrt{3}\Theta_3(2\tau)\right) \quad (43)$$

and

$$-\frac{\pi i}{3}\theta\left[\frac{2}{5/3}\right](\tau, 0)\left(\zeta_3\Theta_3\left(\frac{\tau}{3}\right)+\bar{\zeta}_3\Theta_3\left(\frac{2\tau}{3}\right)-i\bar{\zeta}_3\sqrt{3}\Theta_3(\tau)+i\zeta_3\sqrt{3}\Theta_3(2\tau)\right), \quad (44)$$

the theta derivative $\theta\left[\frac{2}{3}\right]'(\tau, 0)$ is $-\frac{\pi i}{3}\theta\left[\frac{2}{3}\right](\tau, 0)$ times

$$2\zeta_3\Theta_3\left(\frac{4\tau}{3}\right)-\zeta_3\Theta_3\left(\frac{\tau}{3}\right)+\bar{\zeta}_3\Theta_3\left(\frac{2\tau}{3}\right)-2i\bar{\zeta}_3\sqrt{3}\Theta_3(4\tau)+i\bar{\zeta}_3\sqrt{3}\Theta_3(\tau)+i\zeta_3\sqrt{3}\Theta_3(2\tau), \quad (45)$$

and $\theta\left[\frac{2}{4/3}\right]'(\tau, 0)$ equals $-\frac{\pi i}{3}\theta\left[\frac{2}{4/3}\right](\tau, 0)$ times

$$2\bar{\zeta}_3\Theta_3\left(\frac{4\tau}{3}\right)-\bar{\zeta}_3\Theta_3\left(\frac{\tau}{3}\right)+\zeta_3\Theta_3\left(\frac{2\tau}{3}\right)+2i\zeta_3\sqrt{3}\Theta_3(4\tau)-i\zeta_3\sqrt{3}\Theta_3(\tau)-i\bar{\zeta}_3\sqrt{3}\Theta_3(2\tau). \quad (46)$$

Proof. We begin by setting $\varepsilon = \frac{1}{3}$ and $\delta = \frac{1}{3}$ (resp. $\delta = \frac{5}{3}$), so that $\beta = \zeta_3$ (resp. $\beta = \bar{\zeta}_3$) in Equation (9). The constant coefficient is $\frac{\pi i}{3}$, while the other terms are $2\pi i\zeta_3^l q^{l(3n-2)/3}$ (resp. $2\pi i\bar{\zeta}_3^l q^{l(3n-2)/3}$) and $-2\pi i\bar{\zeta}_3^l q^{l(3n-1)/3}$ (resp. $2\pi i\zeta_3^l q^{l(3n-1)/3}$). If the exponent is $\frac{N}{3}$ then N is divisible by 3 if and only if l is, and the sum over these terms was evaluated for Equation (33) in Theorem 3.1 (but with the variable τ divided by 3). For the rest, we decompose $2\pi i\zeta_3^l$ (resp. $2\pi i\bar{\zeta}_3^l$) as the sum of $\pm\pi i(\zeta_3^l - \bar{\zeta}_3^l)$ and $\pi i(\zeta_3^l + \bar{\zeta}_3^l)$, while $-2\pi i\bar{\zeta}_3^l$ (resp. $-2\pi i\zeta_3^l$) is the difference of these expressions. Here the $+$ in the \pm stands for the case with $\delta = \frac{1}{3}$, and the $-$ arises from $\delta = \frac{5}{3}$ (in parentheses). The first terms are just those appearing in part (iv) of Lemma 1.4, so that we can use Equation (31) from Theorem 3.1, but combined with part (i) of Corollary 1.6 (since the last arguments involved only exponents N that are not divisible by 3) for evaluating this contribution. On the other hand, the fact that $2\Re\zeta_3^l = -1$ wherever l is not a multiple of 3 implies that the second terms also combine to the sum from part (iv) of Lemma 1.4, but involving the sign $\left(\frac{N}{3}\right)$. However, the proof of part (i) of Corollary 1.6 (or even that of part (iii) of Proposition 1.3 itself) implies that the total coefficient vanishes if $N \equiv 2 \pmod{3}$, so that omitting this sign $\left(\frac{N}{3}\right)$ does not alter the total power series. Applying part (i) of Corollary 1.6 once more, we establish Equations (39) and (40), since $1 \mp i\sqrt{3}$ is just $2\zeta_3^{\mp 1}$ while $3 \mp i\sqrt{3}$ is the product of the complex conjugate of the latter number and $\mp i\sqrt{3}$.

Still in the case where $\varepsilon = \frac{1}{3}$, but now with $\delta = \frac{2}{3}$ (resp. $\delta = \frac{4}{3}$), we get that $\beta = \zeta_6$ (resp. $\beta = \bar{\zeta}_6$), so that every ζ_3 in the above paragraph has to be replaced by ζ_6 . The part involving powers of $q^{1/3}$ that are divisible by 3 is now described by Equation (34) in Theorem 3.1, while the evaluation of the series arising from the sums involving $\zeta_6^l - \bar{\zeta}_6^l$ can be evaluated as in Equation (35) in that Theorem (combined with part (i) of Corollary 1.6 once again). Observing that $2\Re\zeta_6^l$ coincides with $2\Re\zeta_3^l$ for even l but with minus that number for odd l (i.e., the quotient is $(-1)^l$), we can use part (iii) of Corollary 1.6 for evaluating this series, but we have to invert the series with the doubled argument because of the sign $(\frac{N}{3})$, and we have to apply part (i) of Corollary 1.6 since again this part concerns only terms $q^{N/3}$ with non-integral $\frac{N}{3}$. By writing the resulting global coefficients in terms of ζ_3 and $\sqrt{3}$ as in the previous paragraph, Equations (41) and (42) take their asserted forms.

Back in the case where δ as $\frac{1}{3}$ (resp. $\frac{5}{3}$) and β is ζ_3 (resp. $\bar{\zeta}_3$), but now with $\varepsilon = \frac{2}{3}$, the constant term becomes $\frac{2\pi i}{3}$, and in the non-constant terms (with ζ_3^l and $-\bar{\zeta}_3^l$ once more) the exponents $\frac{3n-2}{3}$ and $\frac{3n-1}{3}$ are replaced by $\frac{6n-5}{6}$ and $\frac{6n-1}{6}$ respectively. As this means halving the value of τ but taking only divisors l with odd $\frac{N}{l}$, the part involving indices l (or equivalently N) that are divisible by 3 is evaluated (again with a different value of τ) in Equation (37) in Theorem 3.1. The terms involving $\zeta_3^l - \bar{\zeta}_3^l$ with l (and N) not divisible by 3 give coefficients that are based on divisors $d = l$ with odd $\frac{N}{d}$. A series with such coefficients was evaluated for Equation (32) in Theorem 3.1, while the restriction on exponents not divisible by 3 requires another application of part (i) of Corollary 1.6. The part with $\zeta_3^l + \bar{\zeta}_3^l = -1$ (for l not divisible by 3) involves again a divisor sum that is restricted by $\frac{N}{d}$ being odd, so that we can apply parts (i) and (ii) of Corollary 1.6 once more, but we have to invert the sign of $\Theta_3(\frac{2\tau}{3})$ and $\Theta_3(2\tau)$ because of the sign $(\frac{N}{3})$. Gathering all the terms together yields Equations (43) and (44).

Finally, let ε remain $\frac{2}{3}$, and take $\delta = \frac{2}{3}$ (resp. $\delta = \frac{4}{3}$) and $\beta = \zeta_6$ (resp. $\delta = \bar{\zeta}_6$) again, and replace every ζ_3 by ζ_6 in the previous paragraph. The series involving indices l (or exponents N) that are divisible by 3 is (up to the usual rescaling of the variable τ) the one appearing in Equation (38) in Theorem 3.1, and the one arising from the differences $\zeta_6^l - \bar{\zeta}_6^l$ with N not divisible by 3 is evaluated using the combination of the (rescaled) Equation (36) in that Theorem and part (i) of Corollary 1.6. In the remaining sum we again replace $\zeta_6^l + \bar{\zeta}_6^l$ by the same expression with ζ_3 but multiplied by $(-1)^l$, but the oddity of $\frac{N}{l}$ implies that this sign is the same as $(-1)^N$. Part (i) of Corollary 1.6 shows how the operation of this multiplication by $(-1)^N$ does to the series, that was otherwise evaluated in the previous paragraph. Collecting terms and writing the coefficients using ζ_3 etc. as in the previous cases, we get Equations (45) and (46). This completes the proof of the theorem. \square

We conclude by remarking that for characteristics with 5 in the denominator we do not have a good description for the associated theta function, since the

class number of $\mathbb{Q}(\sqrt{-5})$ is not 1. Turning to 7 in the denominator, $\mathbb{Q}(\sqrt{-7})$ is of class number 1, but since sums of 7th roots of unity with characters or restricting to only 2 residues modulo 7 (with opposite signs) does not immediately give the required numbers, the evaluation in this case is again substantially more difficult. It is possible though that allowing 6 in the denominator, or mixing 2 and 3 in the denominators of the two characteristics, may still yield expressions that are possible to evaluate. However, these questions are left for future research.

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